B.I.E.M. IN 3 - D ELASTO - PLASTIC PROBLEMS

García Benitez, F.[+]; Gómez Lera, S.; Doblaré, M. & Alarcón, E.

Polytechnical University of Madrid (Spain)

+ Polytechnical University of Sevilla (Spain)

SUMMARY

In this paper we present the application of BIEM to elasto-plastic axysimetric problems. After a brief presentation of the basic integral formulation we introduce the discretizing and ite - rative process for its resolution. Simple problems are compared in order to test the possibilities of the method and we finish commenting on future research needs.

ELASTOPLASTIC FORMULATION WITH SINGULAR INTEGRALS

As is well known BIEM is obtained through the use of a reciprocal relationship and the existence of the operator fundamental solution. For the elastic case the Somigliana identity is obtained when the Maxwell - Betti theorem is used

$$\int \sigma^{*} \varepsilon^{e} = \int \sigma \, \varepsilon^{*} \qquad (1)$$

where the superindex is used merely as a matter of convenience as it will seen soon.

But
$$\int_{\Omega} \sigma \, \varepsilon^{*} = \int_{\Omega} X \, u^{*} + \int_{\partial\Omega} T^{\nu} u^{*} \qquad (2)$$

where X and T^{ν} are the body forces and surfaces tractions, respectively, equilibrating σ . u^{*} is a displacement field compatible with the boundary conditions.

In the plasticity case it is normal to work with increments and then

$$\int_{\Omega} \sigma^{*} \dot{\varepsilon}^{e} = \int_{\Omega} \dot{\sigma} \, \varepsilon^{*} = \int_{\Omega} \dot{X} \, u^{*} + \int_{\partial\Omega} \dot{T}^{\nu} u^{*} \qquad (3)$$

But if it is possible to separate the elastic and plastic components of deformation

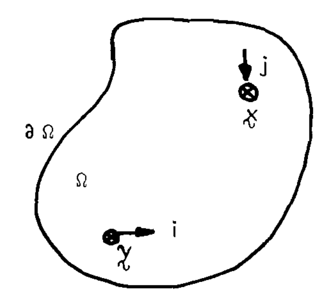
$$\dot{\varepsilon} = \dot{\varepsilon}^{e} + \dot{\varepsilon}^{p} \qquad (4)$$

and
$$\int_{\Omega} \sigma^{*} \dot{\varepsilon}^{p} - \int_{\Omega} \dot{\sigma}^{*} \dot{\varepsilon} = \int_{\Omega} \dot{X} \, u^{*} + \int_{\partial\Omega} \dot{T}^{\nu} u^{*} \qquad (5)$$

As ε is compatible and σ is in equilibrium with X and T^{ν} we can write:

$$\int_{\Omega} X^{*} \dot{u} + \int_{\partial\Omega} T^{\nu *} \dot{u} - \int_{\Omega} \sigma^{*} \dot{\varepsilon}^{p} = \int_{\Omega} \dot{X} \, u^{*} + \int_{\partial\Omega} \dot{T}^{\nu} u^{*} \qquad (6)$$

which is the basic relationship of the method.



$$X^{*}_{i}(x) = \delta_{ij}(y - x) . e_{j}$$

$$T^{\nu *}_{i}(y) = T_{ji}(x, y) . e_{j}$$

$$u^{*}_{i}(y) = U_{ji}(x, y) \, e_{j}$$

$$\sigma_{lk}(y) = \sigma_{jlk}(x, y) . e_{j}$$

$$x, y \in \Omega \qquad (7)$$

Figure 1

If the fundamental solution (Kelvin load) is introduced now, equations (7) are obtained, where x is the application point of a unitary load pointing along 'j' and y is the observation point along 'i' direction.

The substitution of (7) in (6) allows the writing of

$$u_{j}(x) + \int_{\partial\Omega} T_{ji} \dot{u}_{i} = \int_{\Omega} \dot{X}_{i} \dot{U}_{ji} + \int_{\partial\Omega} \dot{T}^{\nu}_{i} U_{ji} + \int_{\Omega} \sigma_{ljk} \dot{\varepsilon}_{lk}$$

$$x, y \in \Omega \qquad (8)$$

that is the generalitazion to the elastoplastic case of Somigliana's relationship. Note that the different kernels depends on the distance $x - y$, being singular when $x \equiv y$ and then the integrals have to be understood in the sense of their principal values.

It is easy to show that the boundary representation formula is :

$$c_{ij}\, \dot{u}_j(\underset{\sim}{x}) + \int_{\partial\Omega} T^{\nu}_{ji}\, \dot{u}_i = \int_{\partial\Omega} \dot{T}^{\nu}_i\, U_{ji} + \int_{\Omega} X_i\, U_{ji} + \int_{\Omega} \sigma_{jik}\, \varepsilon^{p}_{ik}$$

$$\underset{\sim}{x} \in \partial\Omega \qquad (9)$$

where c_{ij} depends on boundary geometry.

As can be seen the last two integrals are extended to the whole Ω. When X can be derived from a potential function (self weight, centrifugal forces, termoelastic case, etc.). Neverthe less in the elastoplastic case, the last integral is unavoidable , being neccesary to compute the internal stress. As it is well known they are obtained from

$$\dot{\sigma}_{ij} = 2\, G\, \dot{\varepsilon}_{ij} + \delta_{ij} \frac{2\, G \nu}{1 - 2\nu} \dot{\varepsilon}_{kk} - 2\, G\, \dot{\varepsilon}^{p}_{ij} \qquad (10)$$

where

$$\dot{\varepsilon}_{ij} = (\dot{u}_{i,j} - \dot{u}_{j,i})/2$$

Using the representation formula (8)

$$\dot{\sigma}_{ij} = \int_{\partial\Omega} D_{kij}\, \dot{T}_k - \int_{\partial\Omega} S_{kij}\, \dot{u}_k + \int_{\Omega} D_{kij}\, \dot{X}_k -$$

$$- \frac{G}{4(1-\nu)} \left(2\, \dot{\varepsilon}^{p}_{ij} + (1 - 4\nu)\, \dot{\varepsilon}^{p}_{ll}\, \delta_{ij} \right) +$$

$$+ \int_{\Omega} \sigma_{ijkl}\, \dot{\varepsilon}_{kl} \qquad (11)$$

where the last two terms correspond to material plastification.

The computation of the last integral has been repetedly misinterpreted before 1978 when H. D. Bui got the correct solu - tion applying the concept of integral derivative developed by -- Mikhlin.

In summary, the rest of equations to be solved are (8), (9) and (10), what implies a formidable analytic task only manageable through the use of a numerical procedure.

THE BIEM APPROACH

For the solution of the previous equation a projective method of the internal class in Temamś sense is used. It is characterized by:

a) A Triangulation $\mathcal{C}_h$ on $\partial\Omega$. That is, a series of bounda- ry elements $K \in \mathcal{C}_h$ whose union reproduces $\partial\Omega$.

b) The displacements and stress fields are introduced in the boundary by polynomial pieces, in the sense that for each $K \in \mathcal{C}_h$ the spaces

$$P_k = (v_k /_k \;\; ; v_h \in V_h)$$

are constituted by polinomials.

c) The approximating space is generated through bases of plynomial splines of compact support.

The triangulation properties are similar to those stablished by CIARLET (1978) for the F.E.M., that is:

1) $\partial\Omega = \bigcup_{k \subset \mathcal{C}_h} K$

2) $\forall K_1 , K_2 \subset \mathcal{C}_h \; , \; \mathring{K}_1 \cap \mathring{K}_2 = \phi$

3) $\forall K \in \mathcal{C}_h$, K is closed and its interior $\mathring{K}$ empty

4) $\forall K \in \mathcal{C}_h$, ∂K is Lipschitz - continuous

In the plastic case to those "boundary elements " we add volumetric tridimensional "cells" (generally tethraedrous) with similar properties although their use is limited to the computation of integrals.

The functional approximation is of the usual form

$$u \simeq u_n = \sum_1^n a_j\, \Psi_j \qquad (12)$$

where a_j are coefficients to be determined and Ψ_j the polynomial splines we referred to above. The fundamental difference with F.E.M. are the projection functions which are the response to the fundamental solution and which produce a collocation method. As a consequence the matrices are asimetric and full, because the fundamental solutions produce effects everywhere round the boundary.

After the discretization, equations (9) and (10) can be written as

$$A \dot{x} = f + D \dot{\varepsilon}^{p} \qquad \text{(a)}$$

$$\dot{\sigma} = - A' \dot{x} + f' + D \dot{\varepsilon}^{p} \qquad \text{(b)} \qquad (13)$$

Premultiplying (a) by A^{-1}

$$\dot{x} = A^{-1} f + A^{-1} D \dot{\varepsilon}^{p} = m + K \dot{\varepsilon}^{p}$$

and entering (13 b)

$$\dot{\sigma} = - A' (m + K \dot{\varepsilon}^{p}) + D \dot{\varepsilon}^{p} = - A' m + (D - A' K) \dot{\varepsilon}^{p}$$

or

$$\sigma = n + B \dot{\varepsilon}^{p} \qquad (14)$$

m and n are the elastic part of the solution.

The incremental equations are then

$$\dot{X} = m + K \dot{\varepsilon}^{p}$$

$$\dot{\sigma} = n + B \dot{\varepsilon}^{p} \qquad (15)$$

that after summing-up give for time j

$$X_j = m + K (\dot{\varepsilon}^{p}_{j-1} + \dot{\varepsilon}^{p}_{j})$$

$$\sigma_j = n + B (\dot{\varepsilon}^{p}_{j-1} + \dot{\varepsilon}^{p}_{j}) \qquad (16)$$

ITERATION PROCESS

The solving process is an step - by - step one. Inside each iteration the involved matrices are assumed constant while a certain error bound is exceeded.

If has been traditional to use one of the methods schematized in Figure 2 for the Von Mises case

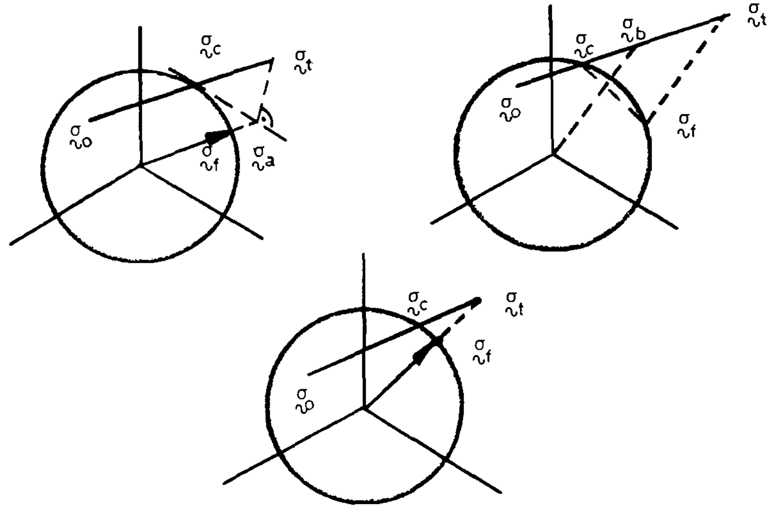


Figure 2

The first is the tangent prediction and radial correction, the second is the RICE and TRACY method and the third is the elastic prediction and radial correction developed after a Mendelson idea. We have used the last one just because of its simplicity and its accuracy comparable to the other two methods, as demonstrated by Krieg and Schreyer et al.

For the isotropic Von Mises case

$$f (s, \lambda) = s_{ij} \, s_{ij} - k^{2} (\lambda) = 0 \qquad (17)$$

the deviator component is taken as

$$\dot{e}^{p} = \dot{\beta} (\partial f / \partial s) \qquad (18)$$

and an experimental relationship of the form

$$k = \sqrt{\frac{2}{3}} \; Y (\lambda) \qquad (19)$$

is admited, where Y is the yield limit for the monodimensional test.

With it

$$\dot{e}^{P} = 2\,\dot{\beta}\, s^{f}_{ij} \tag{20}$$

and using

$$\dot{\lambda} = 2\sqrt{\frac{2}{3}}\,\dot{\beta}\, k_f$$

$$k_f = \sqrt{\frac{2}{3}}\; Y\,(\lambda_o + \dot{\lambda}) \tag{21}$$

As

$$e^{eF}_{ij} = e^{eT}_{ij} - \Delta\,\varepsilon^{p}_{ij} \tag{22}$$

and

$$s^{T}_{ij} = s^{o}_{ij} + 2\,G\,\Delta\, e_{ij} \tag{23}$$

$$s^{F}_{ij} = 2\,G\; e^{eF}_{ij} = 2\,G\, e^{T}_{ij} - 2\,G\,\Delta\,\varepsilon^{p}_{ij} = s^{T}_{ij} - 2\,G\,\Delta\,\varepsilon^{p}_{ij} \tag{24}$$

Using eq (20)

$$s^{F}_{ij} = s^{T}_{ij} - 4\,G\dot{\beta}\, s^{F}_{ij} \tag{25}$$

that is

$$s^{F}_{ij} = \frac{s^{T}_{ij}}{1 + 4\,G\dot{\beta}} \tag{26}$$

$$\bar{\sigma}_T = \sqrt{(3/2)\, s^{T}_{ij}\; s^{T}_{ij}} \tag{27}$$

Remembering (17) and (21)

$$\sqrt{s^{F}_{ij}\; s^{F}_{ij}} = \frac{\sqrt{(2/3)}\;\bar{\sigma}_T}{1+4G\dot{\beta}} = k_F = \sqrt{\frac{2}{3}}\; Y$$

and

$$\dot{\beta} = \frac{1}{4G}\left(\frac{\bar{\sigma}_T}{Y} - 1\right)$$

$$\dot{\lambda} = \frac{\bar{\sigma}_T - Y\,(\lambda_o + \dot{\lambda})}{3\,G} \tag{29}$$

EXAMPLES

In what follows we include two threedimensional examples. ne first one is a cantilever beam under different loads. Figure 3 ɔllects the studied cases while Figure 4 shows the discretizations

The first involves 58 boundary elements and 15 domain cells nile the second is a refinement with 82 and 30 pieces respective-·. In Figure 5 we show the progressive collapse of the beam for different values of the load factor.

Figure 6 is a classical problem solved with F.E.M. in 1970 / CHEN and in 1973 by LEE and KOBAYASHI. As it can be seen is the punching of and elastoplastic layer by a circular rigid – ɔoting. Figure 7a represents a three-dimensional discretization nd Figure 7b the axysimetric one.

In the first one we have used 84 boundary elements and 36 ɔlumetric cells concentrated round the plastificating zone.

Figure 8 shows the results in comparison with the Finite lement one.

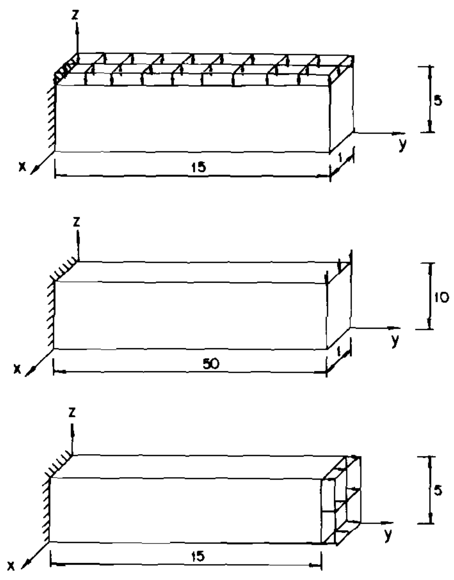


FIG. 3

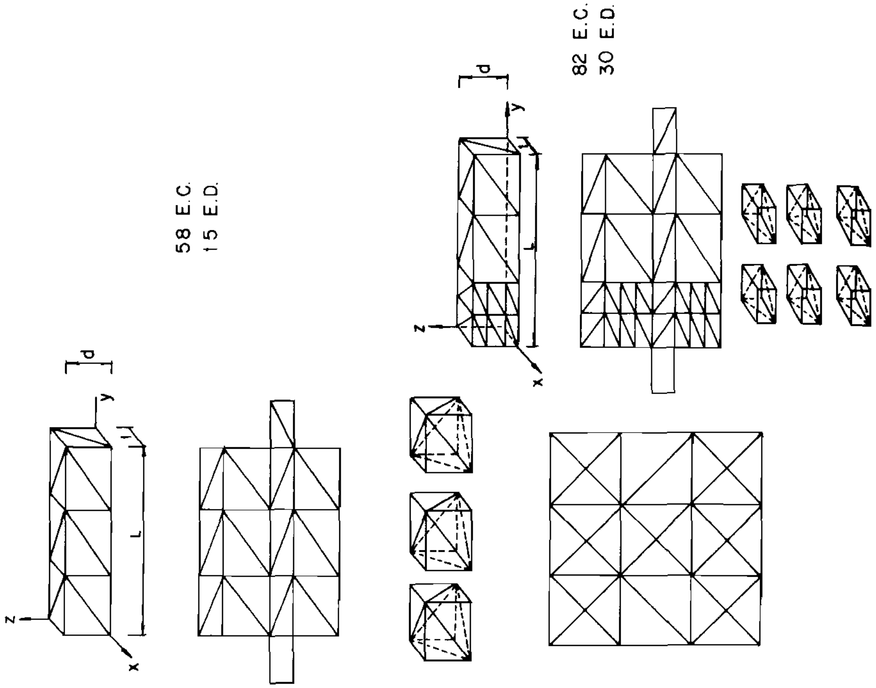


FIG.4

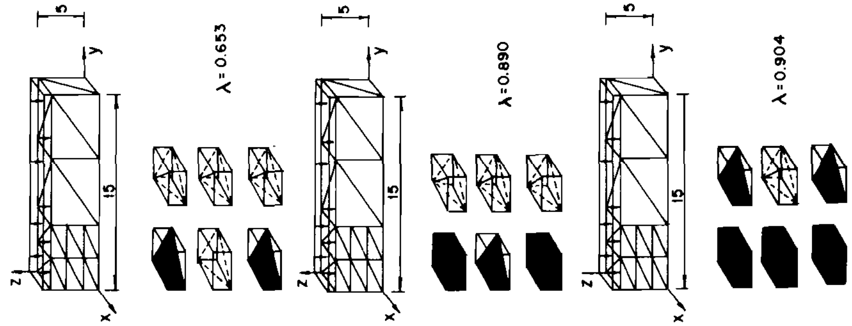
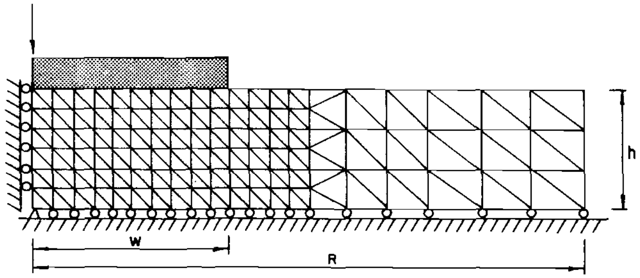


FIG.5



FINITE ELEMENT MESH

FIG.6

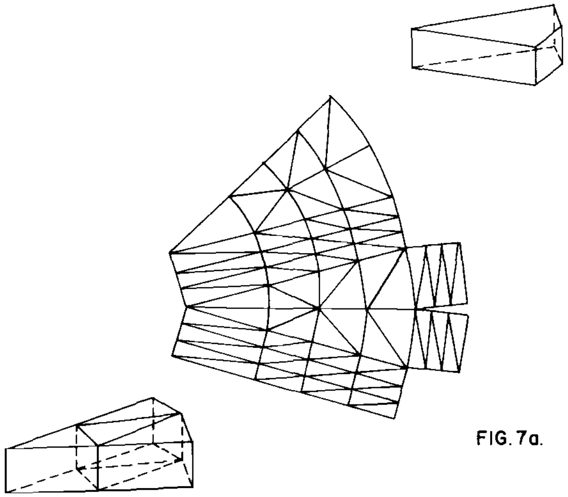


FIG. 7a.

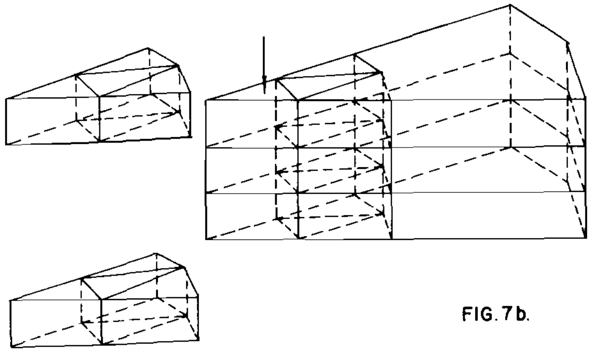


FIG. 7b.

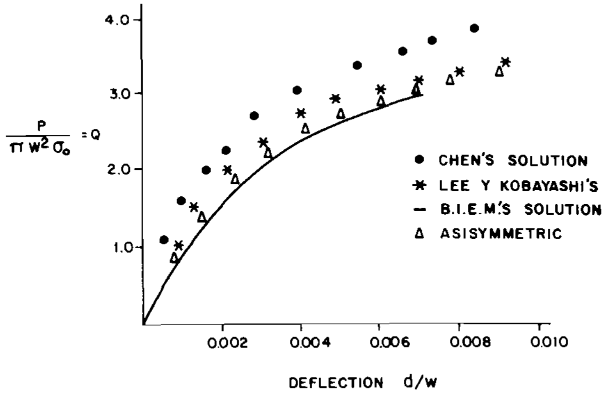
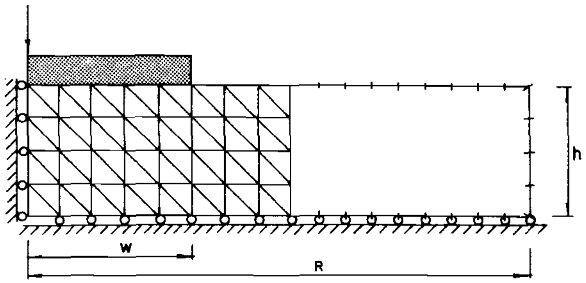


FIG. 8

The latter is composed by 38 elements and 64 cells placed also only near the plastification region. The curve of results - shows a better approximation to the F.E. results than the 3-D one, wich is a common feature for actual axisymmetric problems.



38 ELEMENTS
64 CELLS
B.E.M MESH

FIG. 9

FUTURE TRENDS

The method is still in an experimental and comparative phase. The following steps are needed:

1.- Effective methods to integrate and to solve the system of equations.

2.- Automation of · iterative process.

3.- Better definition of the deformations.

4.- Incorporation of the cyclic plasticity models.

In conclusion it can be said that the BIEM appears as a - powerful tool for elastoplastic computations. Its competitivity in face of FEM is yet debatable due substantially to the amount of time needed for the iterations.

Some alternatives are being testing currently and it is hoped that in the near future it will be possible to fully take advantage of all the best BIEM features.

REFERENCES

1.- BUI, H.D. " Some remarks about the formulation of 3-D thermoelastoplastic problems by Integral Equa tions". Int. J. Solid Structures 14. (1978).

2.- GARCIA BENITEZ, F." Formulación del método de las ecuaciones integrales de contorno en elasto - plasticidad tridimensional ". Tesis Doctoral E.T.S.I. Industriales . Madrid. (1981).

3.- HUTCHINSON, J.W. "Finite strain analysis of elasto- plastic solids and structures. Numerical Solu- tion of nonliner structural problems". ASME AMD, Vol 6. (1973).

4.- KRIEG, R.D., KRIEG, D.B."Accuracies of Numerical solution methods for the elastic - perfectly plastic model.". Transaction of the ASME. November (1977).

5.- LACHAT, J.C."A futher development of the B. I. Technique for elastostatic". Ph.D. Thesis. Southampton Univ. (1975).

6.- MENDELSON, A. " Plasticity : Theory and Application" McMillan (1968).

7.- SCHREYER, H. L. et al. "accurate Numerical Solutions for Elastic - Plastic Models" . Trans. ASME August (1979).